MG26018 Simulation Modeling and Analysis
仿真建模与分析

Lecture 6: Output Analysis I: Single Model

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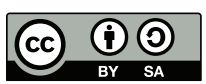

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Introduction

- Output analysis is the examination of data generated by a simulation.
- Output analysis is needed because the output data from a stochastic simulation exhibits random variability
- Suppose the true performance of the *simulated system* is θ.
  - The result from a set of simulation runs will be an estimator $\widehat{\theta}$.
  - The precision of the estimator $\widehat{\theta}$ can be measured by the *standard error* of $\widehat{\theta}$ (i.e., estimator's standard deviation) or the width of a confidence interval for θ.
- The purpose of the statistical analysis:
  - Estimate the true performance θ.
  - Control the estimation precision.
- Types of simulation with regard to output analysis:
  - terminating vs. nonterminating.

- A **terminating simulation** is one that runs for some well-defined time duration T_E.
  - E is a specified event (or set of events) that stops each simulation run (replication).
  - Simulation starts at time 0 under well-specified initial conditions, and ends at the stopping time T_E.
  - T_E can be either a deterministic or random variable.

- Example: A bank opens at 9 AM (*time* 0) with no customers present and 8 of the 11 tellers working (*initial conditions*), and closes at 5 PM (*time* $T_E = 8$ *hours*).
  - $E = \{$8 hours of simulated time have elapsed$\}$.

- It actually stops service when the last customer who entered before 5 PM has been served.
  - $E = \{$at least 8 hours of simulated time have elapsed and the system is empty$\} \Rightarrow T_E$ is a random variable.

- A **nonterminating simulation** is one that runs continuously and without a natural event E to stop the simulation run.
  - Initial conditions are defined by the analyst, but its effect fades away as simulation time increases.
  - Stopping time is conceptually infinite, and in practice it is determined by the analyst with certain statistical precision.
- Examples: Production line that runs 24/7, hospital emergency rooms, continuously operating computer networks, etc.
- For a simulation model that is run in a nonterminating way and has a steady-state (stationary) distribution:
  - The objective is often to study the long-run, or steady-state, behavior of a system, which is not influenced by the initial conditions.
  - Such nonterminating simulation is also called *steady-state simulation*.

- Suppose we want to estimate $\theta = \mathbb{E}[X]$ based on the iid sample $\{X_1, \ldots, X_n\}$.
- The point estimator (a single variable) is

$$\widehat{\theta} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

- How good is this estimator?
  - Unbiased: $\mathbb{E}[\widehat{\theta}] = \theta$.
  - Consistent: $\widehat{\theta} \to \theta$ almost surely as $n \to \infty$.[†]
- Point estimator says *nothing* about the estimation error for finite sample size n.
  - Small estimation error means high estimation precision.

[†] Assume $\mathbb{E}[|X|] < \infty$, or $\sigma^2 < \infty$.

- If $X \sim \mathcal{N}(\theta, \sigma^2)$, then

$$\sqrt{n}\left(\frac{\widehat{\theta} - \theta}{\sigma}\right) \sim \mathcal{N}(0, 1).$$

- If X follows arbitrary distribution and $\sigma^2 = \mathrm{Var}(X) \in (0, \infty)$, then by Central Limit Theorem,

$$\sqrt{n}\left(\frac{\widehat{\theta} - \theta}{\sigma}\right) \Rightarrow \mathcal{N}(0, 1), \quad \text{as } n \to \infty. \tag{1}$$

  - $\sqrt{n}\left(\frac{\widehat{\theta}-\theta}{\sigma}\right) \sim \mathcal{N}(0, 1)$ approximately when n is large.

- σ^2 is typically unknown, and we substitute it by the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

- If $X \sim \mathcal{N}(\theta, \sigma^2)$, then

$$\sqrt{n}\left(\frac{\widehat{\theta} - \theta}{S}\right) \sim t_{n-1}, \tag{2}$$

  where t_p denotes t distribution with p degrees of freedom.

- If X follows arbitrary distribution and $\sigma^2 = \mathrm{Var}(X) \in (0, \infty)$, then by Equation (1) and the fact that $\frac{\sigma}{S} \to 1$ almost surely as $n \to \infty$,

$$\sqrt{n}\left(\frac{\widehat{\theta} - \theta}{S}\right) \Rightarrow \mathcal{N}(0, 1), \quad \text{as } n \to \infty. \tag{3}$$

  - $\sqrt{n}\left(\frac{\widehat{\theta}-\theta}{S}\right) \sim \mathcal{N}(0, 1)$ approximately when n is large.

- Results (2) and (3) are the basis of the confidence interval estimation for θ.

- If $X \sim \mathcal{N}(\theta, \sigma^2)$, where θ and σ are unknown, then a $1-\alpha$ confidence interval (CI) for θ is

$$\left[\widehat{\theta} - t_{n-1,1-\alpha/2}\frac{S}{\sqrt{n}},\ \widehat{\theta} + t_{n-1,1-\alpha/2}\frac{S}{\sqrt{n}}\right], \tag{4}$$

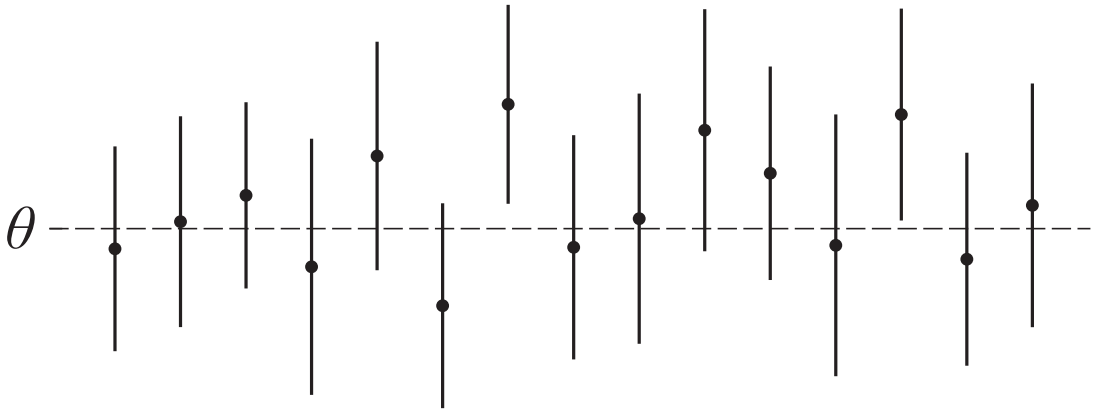
where $t_{n-1,1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of t_{n-1} dist.

Proof.

$$\begin{aligned}
&\mathbb{P}\left\{\theta \in \left[\widehat{\theta} - t_{n-1,1-\alpha/2}S/\sqrt{n},\ \widehat{\theta} + t_{n-1,1-\alpha/2}S/\sqrt{n}\right]\right\} \\
=&\mathbb{P}\left\{|\theta - \widehat{\theta}| \le t_{n-1,1-\alpha/2}S/\sqrt{n}\right\} \\
=&\mathbb{P}\left\{\left|\frac{\theta - \widehat{\theta}}{S/\sqrt{n}}\right| \le t_{n-1,1-\alpha/2}\right\} = 1-\alpha,
\end{aligned}$$

where the last equality is due to (2) and the symmetry of t distribution. ■

- The interpretation of CI:
  - If one constructs a very large number of independent $1-\alpha$ CIs, each based on n observations, the proportion of CIs that actually contain (cover) θ should be $1-\alpha$.



  - **Caution**: There is nothing probabilistic about a single CI after the data have been observed and the interval's endpoints have been given numerical values, e.g., $[1.1,\ 2.4]$.

- Try it out! `http://www.rossmanchance.com/applets/ConfSim.html`

- If X follows arbitrary distribution, θ and $\sigma^2 = \mathrm{Var}(X)$ are unknown, and $0 < \sigma^2 < \infty$, then an approximate $1-\alpha$ CI for θ with large n is

$$\left[\widehat{\theta} - z_{1-\alpha/2}\frac{S}{\sqrt{n}},\ \widehat{\theta} + z_{1-\alpha/2}\frac{S}{\sqrt{n}}\right], \tag{5}$$

  where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of $\mathcal{N}(0,1)$.
- The proof is similar as before by using (3), instead of (2).
- The interpretation is the same as before.
- In practice, people also use (4) as approximate CI even when X does not follow a normal distribution.
  - Both (4) and (5) are approximation for finite n when X is non-normal.
  - $t_{n-1,1-\alpha/2} > z_{1-\alpha/2}$, so CI (4) will be wider than CI (5).
  - CI (4) generally has coverage closer to the desired level $1-\alpha$.
  - $t_{n-1,1-\alpha/2} \to z_{1-\alpha/2}$ as $n \to \infty$.

- For CI (4), the half-length under $1-\alpha$ confidence level is

$$H = t_{n-1,1-\alpha/2}\frac{S}{\sqrt{n}}.$$

- For CI (5), the half-length under $1-\alpha$ confidence level is

$$H = z_{1-\alpha/2}\frac{S}{\sqrt{n}}.$$

- Half-length H presents the precision (or error) of the estimation for θ.
- We want H to be small enough for our decision making, say, $H \leq \epsilon$, under $1-\alpha$ confidence level.

- Usually we take an initial sample of size n_0 to get an estimate of σ^2, say S_0^2.
  - Assume that the estimate of σ^2 will not change (appreciably) from S_0^2 as the sample size increases.
- For CI (4), an approximate expression for the total sample size required to make $H \le \epsilon$ is given by

$$n^* = \min\left\{ n \ge n_0 : \ t_{n-1,1-\alpha/2}\frac{S_0}{\sqrt{n}} \le \epsilon \right\}.$$

- For CI (5), an approximate expression is given by

$$n^* = \min\left\{ n \ge n_0 : \ z_{1-\alpha/2}\frac{S_0}{\sqrt{n}} \le \epsilon \right\} = \left\lceil \left(\frac{z_{1-\alpha/2} S_0}{\epsilon} \right)^2 \right\rceil. \quad (6)$$

- For simplicity, people sometimes use (6), regardless of the distribution of X.
- Take $n^* - n_0$ additional sample points, or start over and take a sample of size n^*, to form the $1-\alpha$ CI (with new S).

- Suppose an iid sample is taken and the values are as follows:

79.919	3.081	0.062	1.961	5.845	0.941	0.878	3.371	2.157	7.579
3.027	6.505	0.021	0.013	0.123	0.624	5.380	3.148	7.078	23.960
6.769	59.899	1.192	34.760	5.009	0.590	1.928	0.300	0.002	0.543
18.387	0.141	43.565	24.420	0.433	7.004	31.764	1.005	1.147	0.219
144.695	2.663	17.967	0.091	9.003	3.217	14.382	1.008	2.336	4.562

- Construct a 95% CI and a 99% CI for $\theta = \mathbb{E}[X]$.

  $n = 50$, $\widehat{\theta} = \bar{X} = 11.894$, $S = 24.953$. We use CI (4) and get $t_{49,0.975} = 2.010$, $t_{49,0.995} = 2.680$. Then,

  95% CI: $11.894 \pm 2.010 \times \frac{24.953}{\sqrt{50}} = 11.894 \pm 7.093 = [4.801, 18.987]$;

  99% CI: $11.894 \pm 2.680 \times \frac{24.953}{\sqrt{50}} = 11.894 \pm 9.457 = [2.437, 21.351]$.

- Want to make half-length $H \leq 2$ under 95% confidence level.

  We use (6) and get $z_{0.975} = 1.960$, $S_0 = S = 24.953$, $\epsilon = 2$. Then,

  $n^* = \left\lceil \left(\frac{1.960 \times 24.953}{2} \right)^2 \right\rceil = \lceil 597.995 \rceil = 598.$

  Take $598 - 50 = 548$ additional sample points.

Terminating Simulation

- A terminating simulation runs over the time interval $[0, T_E]$ and produce observations (outputs).
- Discrete outputs: $\{Y_1, Y_2, \ldots, Y_n\}$.
  - n may be deterministic or random depending on how T_E is specified.
  - E.g., waiting time of all customers.
  - A common goal is to estimate $\theta := \mathbb{E}[\frac{1}{n}\sum_{i=1}^{n} Y_i]$, e.g., the *expectation* of the *average* waiting time.
- Continuous outputs: $\{Y(t) : \ 0 \leq t \leq T_E\}$.
  - T_E may be deterministic or random.
  - E.g., number of customer in the waiting line at time t, $0 \leq t \leq T_E$.
  - A common goal is to estimate $\theta := \mathbb{E}\left[\frac{1}{T_E}\int_0^{T_E} Y(t)\mathrm{d}t\right]$, e.g., the *expectation* of the *average* waiting line length.
- In general, independent replications (runs) are used, each with a different random number stream.

- Within-replication data vs. across-replication data:

Replication	Within-Rep Data	Across-Rep Data
1	$Y_{11}, Y_{12}, \ldots, Y_{1n_1}$	$\bar{Y}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} Y_{1i}$
2	$Y_{21}, Y_{22}, \ldots, Y_{2n_2}$	$\bar{Y}_2 = \frac{1}{n_2}\sum_{i=1}^{n_2} Y_{2i}$
$\vdots$	$\vdots$	$\vdots$
R	$Y_{R1}, Y_{R2}, \ldots, Y_{Rn_R}$	$\bar{Y}_R = \frac{1}{n_R}\sum_{i=1}^{n_R} Y_{Ri}$

- Across-rep data are independent and identically distributed, when *same* initial conditions and *different* random number streams are used.
- Within-rep data are typically **neither independent nor identically distributed**:
  - waiting times of successive customers are heavily correlated;
  - waiting times during the peak hours are longer than off-peak hours, so they're not identically distributed.
- Use **across-rep** data to do point/interval estimation!

- Example: What is the expectation of average waiting time for customers during $[0, T_E]$?
  - Use $\{\bar{Y}_1, \ldots, \bar{Y}_R\}$ as an iid sample of size R.
- Point estimator:

$$\bar{Y} = \frac{1}{R} \sum_{r=1}^{R} \bar{Y}_r.$$

- $1 - \alpha$ CI using (4):

$$\left[\bar{Y} - t_{R-1,1-\alpha/2} \frac{S}{\sqrt{R}},\ \bar{Y} + t_{R-1,1-\alpha/2} \frac{S}{\sqrt{R}} \right],$$

  where $S^2 = \frac{1}{R-1} \sum_{r=1}^{R} (\bar{Y}_r - \bar{Y})^2$.

- Necessary number of replications for specified precision $H \leq \epsilon$ under $1 - \alpha$ confidence level using (6).

- Within-replication data vs. across-replication data:

Replication	Within-Rep Data	Across-Rep Data
1	$\{Y_1(t): \ 0 \le t \le T_{E_1}\}$	$\tilde{Y}_1 = \frac{1}{T_{E_1}} \int_0^{T_{E_1}} Y_1(t)\mathrm{d}t$
2	$\{Y_2(t): \ 0 \le t \le T_{E_2}\}$	$\tilde{Y}_2 = \frac{1}{T_{E_2}} \int_0^{T_{E_2}} Y_2(t)\mathrm{d}t$
$\vdots$	$\vdots$	$\vdots$
R	$\{Y_R(t): \ 0 \le t \le T_{E_R}\}$	$\tilde{Y}_R = \frac{1}{T_{E_R}} \int_0^{T_{E_R}} Y_R(t)\mathrm{d}t$

- Across-rep data are independent and identically distributed, when *same* initial conditions and *different* random number streams are used.
- Example: What is the expectation of the average waiting line length during $[0, T_E]$?
  - Use $\{\tilde{Y}_1, \ldots, \tilde{Y}_R\}$ as an iid sample of size R, and the rest is similar as before.

Steady-State Simulation

- Consider a single run of a simulation model whose purpose is to estimate a steady-state, or long-run, performance measure of the system.
  - Theoretically speaking, such steady-state performance measure has nothing to do with initial conditions.
- Discrete outputs: $\{Y_1, Y_2, \ldots\}$.
  - A common goal is to estimate $\phi := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i$.
- Continuous outputs: $\{Y(t) : \ t \geq 0\}$.
  - A common goal is to estimate $\phi := \lim_{T_E\to\infty} \frac{1}{T_E} \int_0^{T_E} Y(t)\mathrm{d}t$.
- However, we cannot simulate a system "to infinity" but must stop somewhere.
  - The simulation run length (n or T_E) is a design choice instead of inherently determined by the nature of the problem.

Steady-State Simulation

- The run length in steady-state simulation needs to be *carefully chosen*, with several considerations:
  - bias that is due to artificial or arbitrary initial conditions;
    – can be severe if run length is too short
    – generally decreases as run length increases
  - the desired precision of the point estimator;
    – measured by the *standard error* (i.e., estimator's standard deviation) or confidence interval half-width
  - budget constraints on the time available to execute the simulation.

- The effect of the initial condition is persistent and typically does not vanish after a *finite* time.
  - Unless the initial condition is specified as the "stationary distribution" of the system, which is unknown in general.
- **Fact 1**: Estimators based on a finite-time simulation (finite n or T_E) are biased:

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] \neq \phi, \quad \mathbb{E}\left[\frac{1}{T_E}\int_0^{T_E} Y(t)\mathrm{d}t\right] \neq \phi.$$

- **Fact 2**: The bias cannot be replicated away:

$$\lim_{R\to\infty} \frac{1}{R}\sum_{r=1}^{R} \bar{Y}_r \neq \phi, \quad \lim_{R\to\infty} \frac{1}{R}\sum_{r=1}^{R} \tilde{Y}_r \neq \phi.$$

- With more replications, we get a more "precise" estimate of an incorrect value.
  - The confidence interval is narrower but it is centered at an incorrect position.

- Example of $M/M/1$ queue: `https://xiaoweiz.shinyapps.io/MM1queue`
  - If $\lambda < \mu$, the system is stable and the **steady-state expectation** (or **long-run average**) of waiting time is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i = \frac{\lambda/\mu}{\mu - \lambda}.$$

  - Choosing different initial conditions (in this example, number of customers in station, also known as initial state) gives different looks of sample paths (over finite time period).

- Methods to reduce initialization bias:
  - intelligent initialization;
  - warm-up period deletion;
  - low-bias estimator (advanced topic).

- Initialize the simulation in a state that is more representative of long-run conditions.
- If the system exists, collect data on it and use these data to specify more nearly typical initial conditions:
  - fit a probability distribution to describe the initial state;
  - or, simply use the sample mean as a representative.
- If the system can be simplified enough to make it analytically solvable, e.g. queueing models, use the theoretical solution to initialize the simulation.
  - Solve the simplified model to find the stationary distribution or most likely conditions (e.g., the expected number of customers in a station).
  - This is another important value of those analytically solvable queueing models.

- The impact of the initial condition gradually vanishes as the run length increases.
- So we divide a simulation run into two periods:
  - warm-up period: from time 0 to time T_0;
  - data-collection period: from time T_0 to T_E.

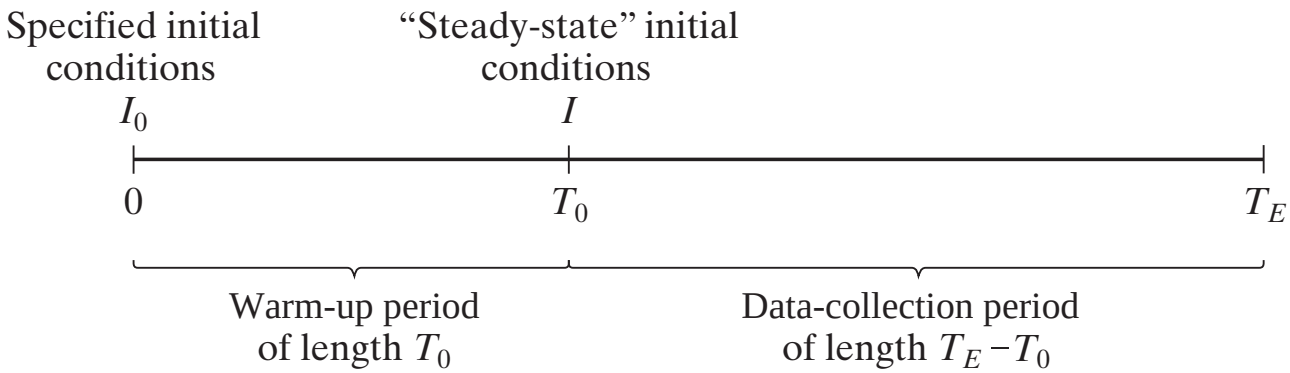


Figure: Warm-up Period Deletion (*from Banks et al. (2010)*)

- T_0 should be sufficiently large so that at time T_0 the impact of the initial condition is very weak and the system behaves approximately as in the steady state.

- To determine T_0
  - There are no widely accepted and proven techniques.
  - Plots are often used.
- The raw output data plot from a single simulation run is usually too variable to detect the trend. – **not helpful**
- Instead of directly plotting raw output data, we usually use some smoother plots to see when the curve "stabilizes":
  - cumulative average (累积均值); – **OK**
  - ensemble average (总体均值). – **recommended**

Figure: Raw Output of Waiting Time of Each Customer in $M/M/1$ Queue with $\lambda = 0.962$ and $\mu = 1$ (*from* ZHANG Xiaowei)

- Cumulative average (累积均值): For one replication, say, replication 1, plot the average from time 0 up to now.
  - Discrete outputs: Plot $\bar{Y}_1(n) = \frac{1}{n}\sum_{i=1}^{n} Y_{1i}$ with respect to n;
  - Continuous outputs: Plot $\tilde{Y}_1(T) = \frac{1}{T}\int_0^T Y_1(t)\mathrm{d}t$ with respect to T.
- It can be plotted for each replication, so we usually detect different warm-up period durations from different replications.
- The cumulative plot is usually conservative, i.e., the warm-up period it detects is longer than necessary.
  - It retains all of the data including the warm-up period, so the bias needs more time to diminish.

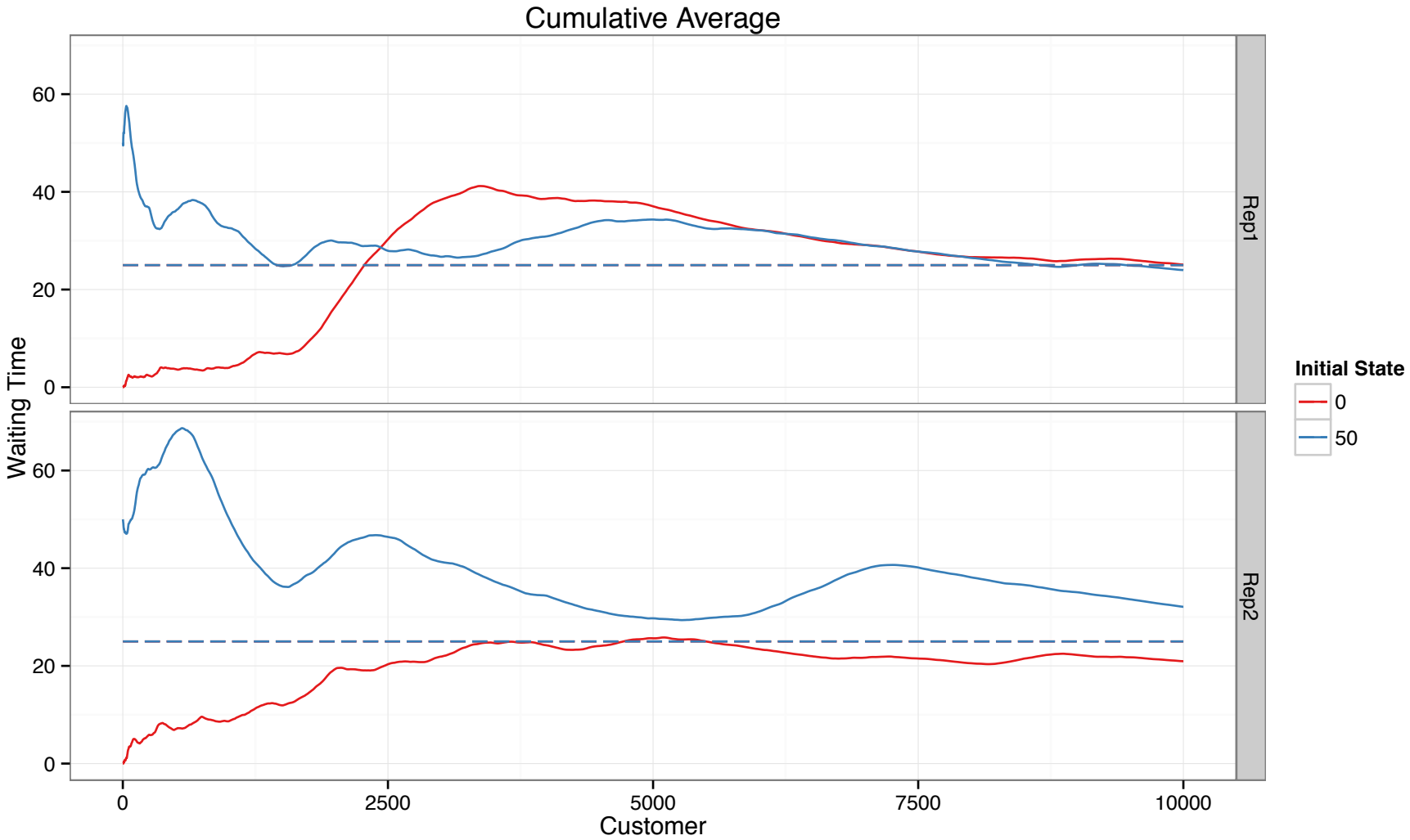


Figure: Cumulative Average Waiting Time of Customers in $M/M/1$ Queue with $\lambda = 0.962$ and $\mu = 1$ (*from* ZHANG Xiaowei)

- Ensemble average (总体均值): For multiple replications $1, \ldots, R$, compute the average across replications and make the plot.
  - Discrete outputs: Plot $\bar{Y}(n) = \frac{1}{R}\sum_{r=1}^{R} Y_{nr}$ with respect to n;
  - Continuous outputs: *Divide the raw data of replication r into small batches*, e.g., $\{Y_r(t) : (j-1)m \le t < jm\}$, $j = 1, 2, \ldots$; plot $\tilde{Y}(j) = \frac{1}{R}\sum_{r=1}^{R}\left[\frac{1}{m}\int_{(j-1)m}^{jm} Y_r(t)\mathrm{d}t\right]$ with respect to j.
- We detect one warm-up period duration for multiple replications.
- Some variations are smoothed out by averaging across multiple replications.
  - This leads to more accurate detection of warm-up period.

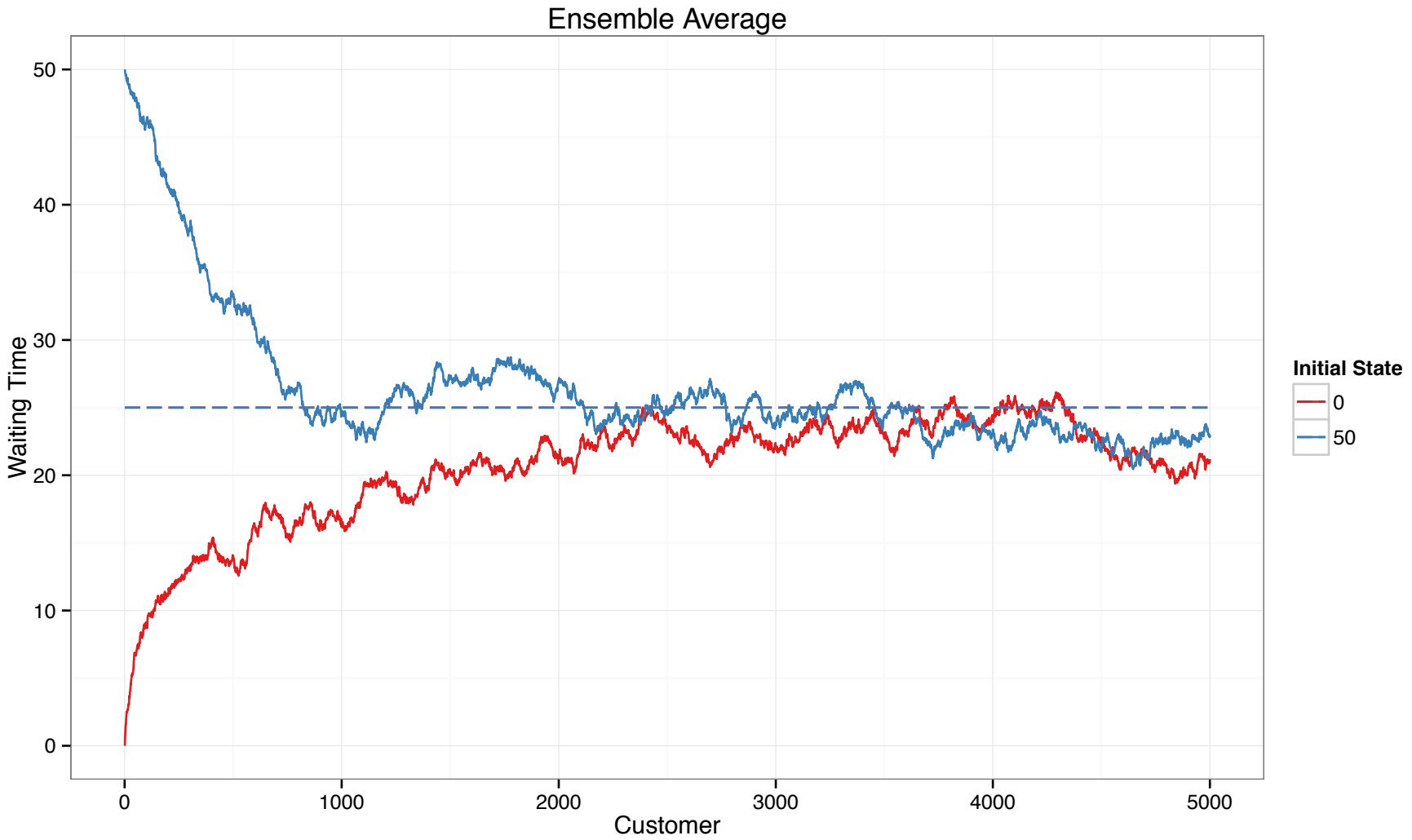


Figure: Ensemble Average Waiting Time of n-th Customer in $M/M/1$ Queue with $\lambda = 0.962$ and $\mu = 1$ (*from* ZHANG Xiaowei)

- When first starting to detect the warm-up period, a run length and number of replications will have to be guessed.
  - Increase the number of replications if the ensemble averages are not smooth enough.
  - Increase the run length if the ensemble averages do not stabilize.
- Since each ensemble average is the sample mean of iid observations across R replications, a confidence interval can be placed around each point.
  - Use them to judge whether or not the plot is precise enough to decide that the bias has vanished.
  - This is the preferred method to determine a deletion point.

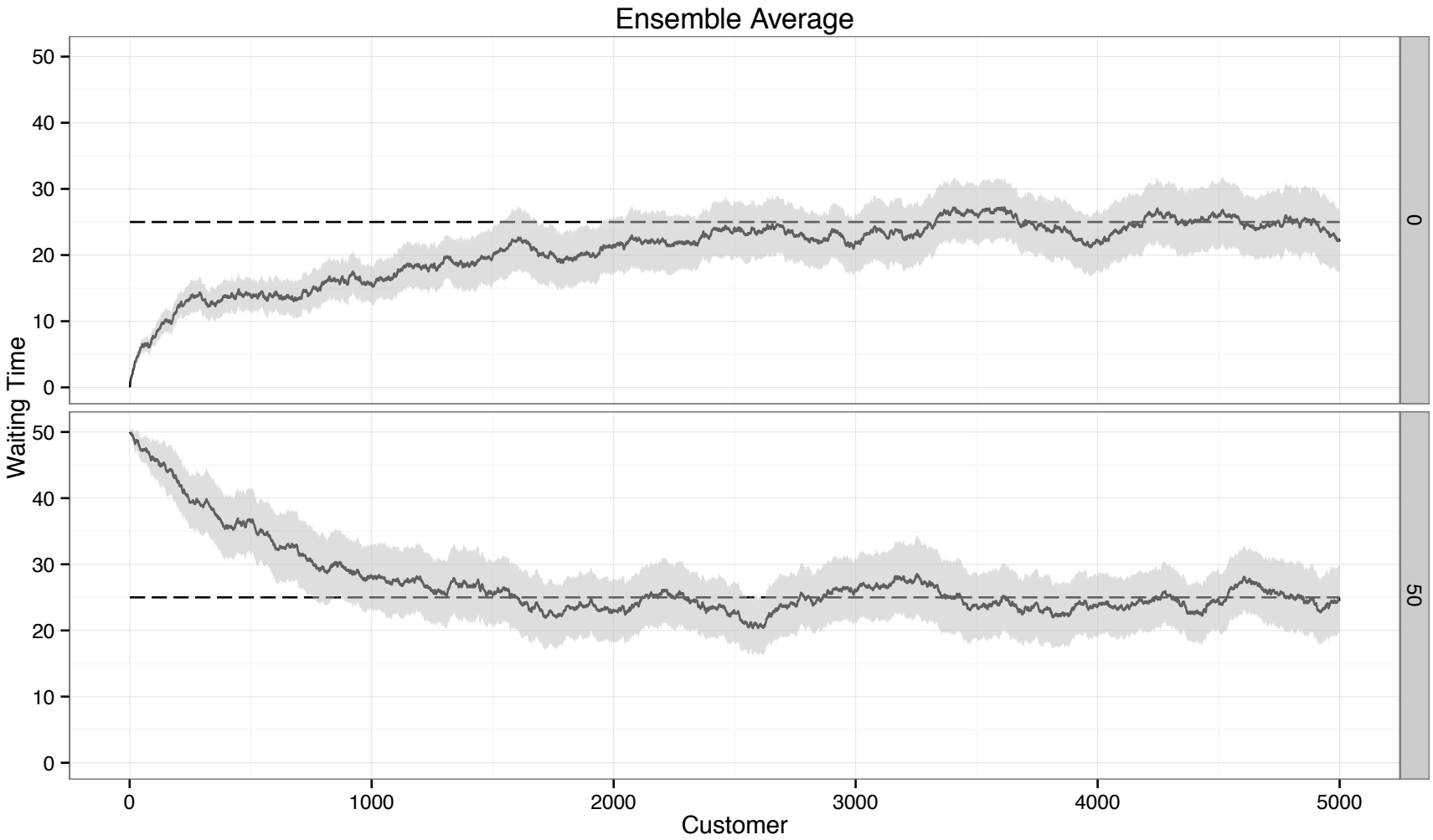


Figure: Ensemble Average Waiting Time and 95% CI of n-th Customer in $M/M/1$ Queue with $\lambda = 0.962$ and $\mu = 1$ (*from ZHANG Xiaowei*)

- Cumulative averages become less variable as more data are averaged.
  - So the right side of the curve is always smoother than the left side.
- Cumulative averages tend to converge more slowly to long-run performance than ensemble averages do.
  - Because cumulative averages contain all observations including the most biased ones from the very beginning.
  - Cumulative averages should be used *only if* ensemble averages can not be computed, such as when only a single replication is possible.
- Different performance measures could approach steady state with different speed.

- Idea: Make multiple replications (long enough), remove warm-up period for each one, and then work as if we were in a terminating simulation.
- Caution: Make sure that initialization bias in the point estimator has been reduced to a negligible level.
  - Otherwise the estimation can be misleading.
- Note: Initialization bias is not affected by the number of replications.
  - It is affected by deleting more data (i.e. increasing T_0) or extending the run length (i.e. increasing T_E).
  - Increasing the number of replications could produce narrower interval around the "wrong point".

- Discrete outputs:
  - Suppose we decide to delete first d observations of the total n observations in a replication.[†]
  - The across-replication data from R replications are

$$\bar{Y}_1 = \frac{1}{n-d}\sum_{i=d+1}^{n} Y_{1i} \,, \ldots, \, \bar{Y}_R = \frac{1}{n-d}\sum_{i=d+1}^{n} Y_{Ri}.$$

- Continuous outputs:
  - Suppose we decide to delete data in $[0, T_0]$ period and only use those in $[T_0, T_E]$ in a replication.
  - The across-replication data from R replications are

$$\tilde{Y}_1 = \frac{1}{T_E - T_0}\int_{T_0}^{T_E} Y_1(t)\mathrm{d}t \,, \ldots, \, \tilde{Y}_R = \frac{1}{T_E - T_0}\int_{T_0}^{T_E} Y_R(t)\mathrm{d}t.$$

[†] d and n may vary between different replications, in which case they are replaced by d_r and n_r, respectively.

- Similar as in terminating simulation, the across-replication data are iid.
  - So we can use them to compute the point estimator, CI, and necessary number of replications for specified precision, in the same way as before.
- Unlike terminating simulation, the above mentioned estimators are biased for finite n or T_E.
  - The bias is negligible if d and n, or T_0 and T_E, are sufficiently large.
- A rough rule for relationship between d and n, or T_0 and T_E:

$$(n-d) \geq 10d, \quad (T_E - T_0) \geq 10T_0.$$

- Suppose analysis indicates that $R - R_0$ additional replications are needed after the initial number R_0, in order to meet the desired precision.
- An alternative to increasing replications is to increase run length T_E within each replication.
  - Increase run length T_E in the same proportion (R/R_0) to a new run length $(R/R_0)T_E$.
  - More data will be deleted, from time 0 to time $(R/R_0)T_0$.
  - More data will be used to compute the estimate, from time $(R/R_0)T_0$ to time $(R/R_0)T_E$.
  - The total amount of simulation effort is the same as if we had simply increased the number of replications.

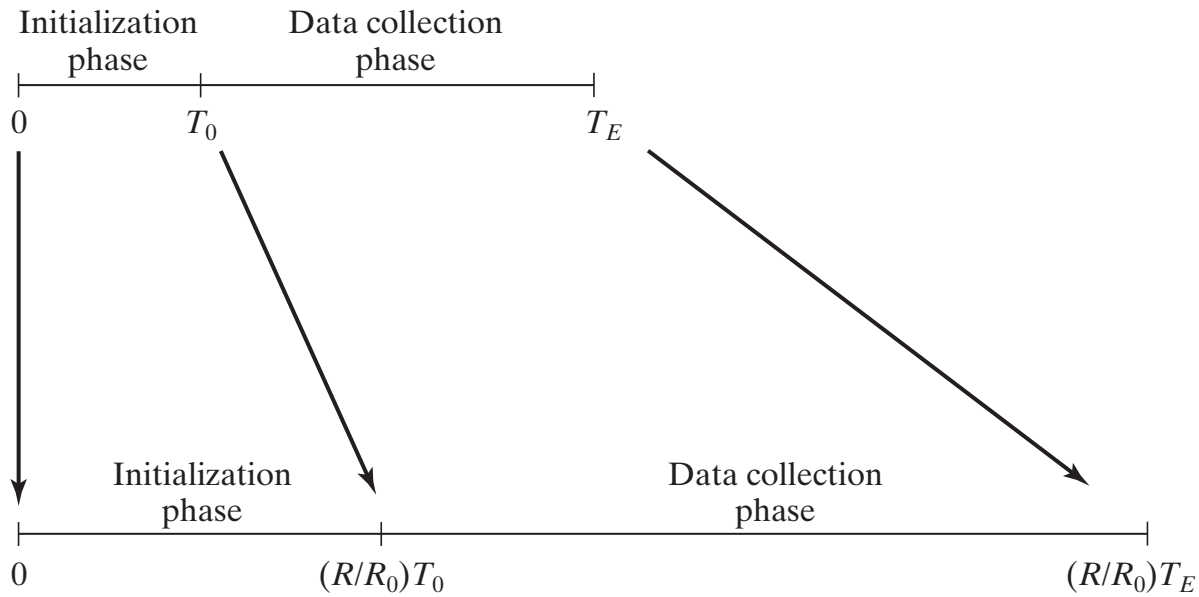


Figure: Increasing Run Length to Achieve Specified Precision (*from Banks et al. (2010)*)

- Advantage: Any residual bias in the point estimator would be further reduced.
- Disadvantage: It is necessary to have saved the state of the model at time T_E and to be able to continue the running.
  - Otherwise, the simulations would have to be re-run from time 0, which could be time consuming for a complex model.

- A disadvantage of the replication method is that the warm-up period must be deleted on each replication.
  - This can become very costly in terms of computation time especially when the model warms up very slowly.
    – E.g., $M/M/1$ queue with utilization close to 1.
- This suggests that we could use one single, (very) long replication for estimation, so that only one warm-up period is deleted.
- Besides, it is also possible that we are in a situation where only the data from one long replication are available.

- Point estimator: Sample mean after the warm-up period deletion

$$\bar{Y} = \frac{1}{n-d} \sum_{i=d+1}^{n} Y_i, \quad \tilde{Y} = \frac{1}{T_E - T_0} \int_{T_0}^{T_E} Y(t) \mathrm{d}t.$$

- The disadvantage of the single-replication design arises when we try to estimate the variance of the above estimators, because of
  - the strong but unknown dependence among $Y_1, Y_2, \ldots, Y_n$;
  - the non-identical distribution of $Y_1, Y_2, \ldots, Y_n$;
  - and the integral form of $\tilde{Y}$.

- **Caution:** It is tempting to compute

$$S^2 = \frac{1}{n-d-1} \sum_{i=d+1}^{n} (Y_i - \bar{Y})^2,$$

and use $S^2/(n-d)$ to estimate $\mathrm{Var}(\bar{Y})$. However, such estimation will be **terrible**, since $Y_1, Y_2, \ldots, Y_n$ are **neither independent nor identically distributed**.

- The CI based on $S^2/(n-d)$ would also be misleading.
- Example: Suppose $Y_{d+1}, \ldots, Y_n$ are identically distributed but positive correlated (which is common for waiting time), then

$$\mathbb{E}[S^2/(n-d)] < \mathrm{Var}(\bar{Y}).$$

  - The constructed CI using $S^2/(n-d)$ will be narrower than the actual valid one.

- Batch Means (批均值) Method:
  - Divide the output data from one replication (after deleting warm-up period) into a few **large batches**, and compute the bath means.
  - Discrete outputs: $\bar{Y}_j = \frac{1}{m}\sum_{i=(j-1)m+1}^{jm} Y_{i+d},\ j = 1, 2, ...$

$$\underbrace{Y_1, \ldots, Y_d}_{\text{deleted}}, \underbrace{Y_{d+1}, \ldots, Y_{d+m}}_{\text{Batch 1: } \bar{Y}_1}, \underbrace{Y_{d+m+1}, \ldots, Y_{d+2m}}_{\text{Batch 2: } \bar{Y}_2}, \ldots, \underbrace{Y_{d+(k-1)m+1}, \ldots, Y_{d+km}}_{\text{Batch } k\text{: } \bar{Y}_k}$$

  - Continuous outputs: $\tilde{Y}_j = \frac{1}{m}\int_{(j-1)m}^{jm} Y(t + T_0)\mathrm{d}t,\ j = 1, 2, ...$
  - Treat the means of these batches **as if** they were independent.
- Why it works?
  - The correlation between two observations decreases as they are farther apart.
  - If the batch size is sufficiently large,

    – most of the observations in a batch will be approximately independent of those in other batches;

    – only those near the end of the batches are significantly correlated.

- Strictly speaking, the batch means are not independent.
- However, if the batch size is sufficiently large, successive batch means will be approximately independent.
- Unfortunately, there is no widely accepted and relatively simple method for choosing an acceptable batch size m (or equivalently, choosing a number of batches k).
- Some general guidelines:
  - In most applications, it is suggested to let $10 \leq k \leq 30$, according to Schmeiser (1982).
  - If the run length is to be increased to attain a specified precision, it is suggested to allow both m and k to grow.